A criterion for transience of multidimensional branching random walk in random environment

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Abstract

We develop a criterion for transience for a general model of branching Markov chains. In the case of multi-dimensional branching random walk in random environment (BRWRE) this criterion becomes explicit. In particular, we show that *Condition L* of Comets and Popov [3] is necessary and sufficient for transience as conjectured. Furthermore, the criterion applies to two important classes of branching random walks and implies that the critical branching random walk is transient resp. dies out locally.

Keywords: branching Markov chains, recurrence, transience, random environment, spectral radius

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1 Introduction

A branching Markov chain (BMC) is a system of particles in discrete time. The BMC starts with one particle in an arbitrary starting position x. At each time each particle is independently replaced by some new particles at some locations according to given stochastic substitutions rules that may depend on the location of the substituted particle. Observe that this model is more general than the model studied in [2], [4], [6], [7], [9], where first the particles branch and then, independently of the branching, move according to an underlying Markov chain. In some sense the behavior of BMC is more delicate than the one of Markov chains: while an irreducible Markov chain is either recurrent or transient, either all or none states are visited infinitely often, this dichotomy breaks down for BMC. Let $\alpha(x)$ be the probability that, starting the BMC in x, the state x is hit infinitely often by some particles. There are three possible regimes: *transient* ($\alpha(x)=0\ \forall x$), *weakly recurrent* ($0<\alpha(x)<1$ for some x) and *strongly recurrent* ($\alpha(x)=1\ \forall x$), compare with Gantert and Müller [4] and Benjamini and Peres [1].

This paper is divided in two parts. First we connect transience with the existence of superharmonic functions, see Theorem 2.1, and give a criterion for transience of BMC in Theorem 2.4. These two criteria are interesting on their own: while the first is stated in terms of superharmonic functions, the second is appropriate to give explicit conditions for transience. In addition, we see that transience does not depend on the whole distributions of the substitution rules but only on

their first moments. There are two important classes of BMC where one can speak of criticality, compare with [4] and [10], [14]. In these cases Theorem 2.1 implies that the critical process is transient resp. dies out locally, compare with Subsection 2.1.

In the second part we follow the line of research of Comets, Menshikov and Popov [2], Comets and Popov [3], Machado and Popov [6], [7] and the author [9] and study transience and recurrence of branching random walk in random environment (BRWRE). In this case we can use the criterion of the first part, Theorem 2.4, to obtain a classification of BRWRE in transient and strong recurrent regimes. In particular, we show that the sufficient *Condition L* for transience of Comets and Popov [3] is necessary, too. Classification results of this type were only known for nearest neighbor BRWRE on $\mathbb{Z}$, [2], and on homogeneous trees, [7]. In addition, we show that transience does not depend on the precise form of the distributions, but only on the convex hull of their support. Such phenomena are known for models of this type, compare with [2], [3], [6], [7], [9]. Our method is quite different from [3] since we don't analyze the process on the level of the particles but use superharmonic functions to describe the process on a more abstract level. The only points where we really deal with *particles* are the proofs of Theorems 2.1 and 3.1. Furthermore, the only point where we need the structure of the lattice $\mathbb{Z}^d$ is where the criterion becomes explicit, see Lemma 3.5. All other arguments immediately apply for BRWRE on Cayley graphs, compare with [9]. In the special case where branching and movement are independent the classification in transience and recurrence is already given in [9].

The obtained classification result is quite interesting facing the difficulty of the corresponding questions for random walks in random environment of a single particle, compare with Sznitman [15] and [16].

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2 Transience and recurrence for general BMC

Let X denote the discrete state space of our process. For every $x \in X$ let

$$\mathcal{V}(x) := \left\{ v(x) = (v_y(x))_{y\in X} : v_y(x) \in \mathbb{N},\ \sum_{y\in X} v_y(x) \geq 1 \right\}$$

be the set of all possible substitution rules. Furthermore, let ω_x be a probability measure on $\mathcal{V}(x)$ and call $\boldsymbol{\omega} := (\omega_x)_{x\in X}$ the environment of our process.

The process is defined inductively: at time $n = 0$ we start the process at some fixed starting position, say o, with one particle. At each integer time the particles are independently substituted as follows: for each particle in $x \in X$ independently of the other particles and the previous history of the process a random element of $v(x) \in \mathcal{V}(x)$ is chosen according to ω_x. Then, the

particle is replaced by $v_y(x)$ offspring particles at y for all $y \in Y$. Thus the BMC is entirely described through the environment $\boldsymbol{\omega}$ on X and is denoted $(X, \boldsymbol{\omega})$. In the definition of $\mathcal{V}(x)$ we demand that the process survives forever: $\sum_{y\in X} v_y(x) \geq 1$ ensures that each particle has at least one offspring. This assumption is made for the sake of a better presentation and to avoid the conditioning on the survival of the process. A key quantity are the first moments of the substitution rules $M := (m(x,y))_{x,y\in X}$, where

$$m(x,y) := \sum_{v\in\mathcal{V}(x)} \omega_x(v) v_y(x)$$

denotes the expected number of offspring sent to y by a particle in x. Let $M^n = (m^{(n)}(x,y))_{x,y\in X}$ be the n-fold convolution of M with itself and set $M^0 = I$, the identity matrix over X. We will always assume that M is irreducible:

General Assumption: Let $\boldsymbol{\omega}$ such that M is irreducible, i.e., for all $x, y \in X$ there exists some k such that $m^{(k)}(x,y) > 0$.

Remark 2.1. Let $P = (p(x,y))_{x,y\in X}$ be the transition kernel of an irreducible Markov chain on X. Then the assumption on irreducibility is fulfilled, if for all $x \in X$ we have $\omega_x(v_y(x) \geq 1) > 0$ for all y with $p(x,y) > 0$.

Remark 2.2. The BMC $(X, \boldsymbol{\omega})$ is a general branching process in the sense of Harris [5] with first moment M : each particle of the general branching process is characterized by a parameter x which describes its position in the state space X.

In order to analyze the process we introduce the following notations. Let $\eta(n)$ be the total number of particles at time n and let $x_i(n)$ denote the position of the ith particle at time n. Denote $\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot \mid x_0(1) = x)$ the probability measure for a BMC started with one particle in x.

We define recurrence and transience for BMC in analogy to [1] and [4]:

Definition 2.1. Let

$$\alpha(x) := \mathbb{P}_x\left(\sum_{n=1}^{\infty}\sum_{i=1}^{\eta(n)} \mathbf{1}\{x_i(n) = x\} = \infty\right) \tag{1}$$

be the probability that x is visited infinitely often. A BMC is *transient,* if $\alpha(x) = 0$ for all $x \in X$, and *recurrent* otherwise. The recurrent regime is divided into *weakly recurrent,* $\alpha(x) < 1$ for some x and *strongly recurrent,* $\alpha(x) = 1$ for all $x \in X$.

The definition of transience and recurrence does not depend on the starting position of the process. In fact, due to the irreducibility, $\alpha(x) > 0$ and $\alpha(x) = 0$ hold either for all or none $x \in X$. This can be shown analogously to [1]. In contrast to the model with independent branching and movement we don't have that $\alpha(x) = 1$ either for all or none x, compare with Example 1 in [3]. Observe that our definition differs in the general setting from the one in [3] but coincide in the case of branching random walk in random environment, compare with Theorem 3.1.

The following criterion for transience in terms of superharmonic functions is a straightforward generalization of Theorem 3.1 in [4]. We give the proof since it makes clear where the superharmonic functions come into play. In addition, it is the essential point where we work on the *particle level.*

Theorem 2.1. *The BMC $(X, \boldsymbol{\omega})$ is transient if and only if there exists a positive function f, such that*

$$Mf(x) := \sum_y m(x,y) f(y) \leq f(x) \quad \forall x \in X. \tag{2}$$

Proof. In analogy to [4] and [8], we introduce the following modified version of the BMC. We fix some site $o \in X$, which stands for the origin of X. The new process is like the original BMC at time $n=1$, but is different for $n>1$. After the first time step we conceive the origin as *freezing*: if a particle reaches the origin, it stays there forever without being ever substituted. We denote this new process **BMC***. The first moment M^* of BMC* equals M except that $m^*(o,o)=1$ and $m^*(o,x)=0\ \forall x \neq o$. Let $\eta(n,o)$ be the number of frozen particles at position o at time n. We define the random variable ν as

$$\nu := \lim_{n\to\infty} \eta(n,o) \in \mathbb{N} \cup \{\infty\},$$

and write $\mathbb{E}_x \nu$ for the expectation of ν given that we start the process with one particle in x.

We first show that transience, i.e. $\alpha \equiv 0$, implies $\mathbb{E}_o \nu \leq 1$ and hence, due to the irreducibility of M, that $\mathbb{E}_x \nu < \infty$ for all x. We start the BMC in the origin o. The key idea of the proof is to observe that the total number of particles ever returning to o can be interpreted as the total number of progeny in a branching process $(Z_n)_{n\geq 0}$. Note that each particle has a unique ancestry line which leads back to the starting particle at time 0 at o. Let $Z_0 := 1$ and Z_1 be the number of particles that are the first particle in their ancestry line to return to o. Inductively we define Z_n as the number of particles that are the nth particle in their ancestry line to return to o. This defines a Galton-Watson process $(Z_n)_{n\geq 0}$ with offspring distribution $Z \stackrel{d}{=} Z_1$. We have $Z_1 \stackrel{d}{=} \nu$ given that the process starts in the origin o. Furthermore,

$$\sum_{n=1}^{\infty} Z_n = \sum_{n=1}^{\infty} \sum_{i=1}^{\eta(n)} \mathbf{1}\{x_i(n) = o\}.$$

If $\alpha(o)=0$, then $\sum_{n=1}^{\infty} Z_n < \infty$ a.s., hence $(Z_n)_{n\geq 0}$ is critical or subcritical and $\mathbb{E}_o \nu = E[Z] \leq 1$.

In order to show the existence of a superharmonic function it suffices now to check that $f(x) := \mathbb{E}_x \nu > 0$ satisfies inequality (2). For x such that $m(x,o)=0$ it is straightforward to show that even equality holds in (2). If $m(x,o)>0$, we have

$$\begin{aligned} f(x) = \mathbb{E}_x \nu &= \sum_{y \neq o} m(x,y) \mathbb{E}_y \nu + m(x,o) \cdot 1 \\ &\geq \sum_y m(x,y) \mathbb{E}_y \nu \\ &= Mf(x), \end{aligned}$$

since $\mathbb{E}_o \nu \leq 1$.

Conversely, we consider the BMC* with origin o and define

$$Q(n) := \sum_{i=1}^{\eta(n)} f(x_i(n)),$$

where $x_i(n)$ is the position of the ith particle at time n. It turns out that $Q(n)$ is a positive supermartingale, so that it converges a.s. to a random variable Q_∞. We refer to [8] for the technical details. Furthermore, we have

$$\nu(o) \leq \frac{Q_\infty}{f(o)} \tag{3}$$

for a BMC* with origin o started in o. We obtain using Fatou's Lemma

$$\mathbb{E}_o \nu \leq \frac{\mathbb{E}_o Q_\infty}{f(o)} \leq \frac{\mathbb{E}_o Q(0)}{f(o)} = \frac{f(o)}{f(o)} = 1. \tag{4}$$

Hence, the embedded Galton-Watson process $(Z_n)_{n\geq 0}$ is critical or subcritical and dies out since $\mathbb{P}_o(\nu(0) < 1) > 0$.

$\square$

Condition (2) in Theorem 2.1 suggests that the spectral radius of the operator M plays a crucial role in finding a more explicit condition for transience. To pursue this path let us briefly recall some known properties of irreducible kernels and of their spectral radii. An operator $K = (k(x,y))_{x,y\in X}$ on X is an irreducible kernel if $k(x,y) \geq 0$ for all $x,y \in X$ and for all x,y there is some l such that $k^{(l)}(x,y) > 0$, where $K^l = \left(k^{(l)}(x,y)\right)_{x,y\in X}$ is the lth convolution of K with itself.

Definition 2.2. The Green function of K is the power series

$$G(x,y|z) := \sum_{n=0}^{\infty} k^{(n)}(x,y) z^n, \;\; x,y \in X, \;\; z \in \mathbb{C}.$$

Lemma 2.2. *For all $x,y \in X$ the power series $G(x,y|z)$ has the same finite radius of convergence $R(K)$ given by*

$$R(K) := \left(\limsup_{n\to\infty} \left(k^{(n)}(x,y) \right)^{1/n} \right)^{-1} < \infty$$

Proof. The fact that the power series defining the functions $G(x,y|z)$ all have the same radius of convergence follows from a system of Harnack-type inequalities. Due to the irreducibility of K for all $x_1, x_2, y_1, y_2 \in X$ there exist some $l_1, l_2 \in \mathbb{N}$ such that we have $k^{(l_1)}(x_1, x_2) > 0$ and $k^{(l_2)}(y_2, y_1) > 0$. Thus for every $n \in \mathbb{N}$,

$$k^{(n+l_1+l_2)}(x_1, y_1) \geq k^{(l_1)}(x_1, x_2) k^{(n)}(x_2, y_2) k^{(k_2)}(y_2, y_1).$$

Consequently, for every $z \in \mathbb{R}^+$

$$G(x_1, y_1|z) \geq k^{(l_1)}(x_1, x_2) k^{(l_2)}(y_2, y_1) z^{l_1+l_2} G(x_2, y_2|z). \tag{5}$$

It follows that the radius of convergence of $G(x_1, y_1|z)$ is at least that of $G(x_2, y_2|z)$. The fact that $R(K) < \infty$ follows from the irreducibility of K: let $l \in \mathbb{N}$ such that $k^{(l)}(x,x) = \varepsilon > 0$ then $k^{(nl)}(x,x) \geq \varepsilon^n$ for every $n \geq 0$. □

Definition 2.3. The spectral radius of an irreducible kernel K is defined as

$$\rho(K) := \limsup_{n\to\infty} \left(k^{(n)}(x,y)\right)^{1/n} > 0. \tag{6}$$

The following characterization of the spectral radius in terms of t-superharmonic functions is crucial for our classification:

Lemma 2.3.

$$\rho(K) = \min\{t > 0 : \ \exists\, f(\cdot) > 0 \textit{ such that } Kf \leq tf\}$$

Proof. We sketch the proof and refer for more details to [11] or [18]. We write $S^+(K,t)$ for the collection of all positive functions f satisfying $Kf \leq tf$. Observe that a function in $S^+(K,t)$ is either strictly positive or constant equal to 0. In order to construct a base of the cone $S^+(K,t)$ we fix a reference point $o \in X$ and define

$$B(K,t) := \{f \in S^+(K,t) : f(o) = 1\}.$$

If there is some $f \neq 0$ in $S^+(K,t)$ then $\rho(K) \leq t$ since

$$k^{(n)}(x,x) f(x) \leq K^n f(x) \leq t^n f(x).$$

On the other hand, observe that for $t > \rho(K)$ we have

$$f(x) := \frac{G(x, o|1/t)}{G(o, o|1/t)} \in B(K,t).$$

The fact that $B(K, \rho(K)) = \bigcap_{t>\rho(K)} B^+(K,t) \neq \emptyset$ follows now from the observation that for all $t > \rho(K)$ the base $B(K,t)$ is compact in the topology of pointwise convergence. □

We now obtain as an immediate consequence of Theorem 2.1 a classification in terms of the spectral radius of M.

Theorem 2.4. *The BMC $(X, \boldsymbol{\omega})$ is transient if and only if $\rho(M) \leq 1$.*

We can directly show that $\rho(M) < 1$ implies transience. Just observe that $G(x,y|1) < \infty$ equals the expected number of particles visiting y in a BMC started in x and transience follows immediately. Conversely, the fact that $\rho(M) > 1$ implies the recurrence of the BMC can be also seen by dint of the interpretation as a general branching process and the fact that the spectral

radius of an infinite irreducible kernel $\rho(M)$ can be approximated by the spectral radii of finite irreducible kernels. A subset $Y \subset X$ is called irreducible (with respect to M) if the operator

$$M_Y = (m_Y(x,y))_{x,y\in Y} \tag{7}$$

defined by $m_Y(x,y) := m(x,y)$ for all $x,y \in Y$ is irreducible. Notice that in this case $\rho(M_Y)$ is just the Perron-Frobenius eigenvalue of M_Y. We find

$$\rho(M) = \sup_Y \rho(M_Y), \tag{8}$$

where the supremum is over finite and irreducible subsets $Y \subset X$. Therefore, if $\rho(M) > 1$ then there exists a finite and irreducible Y such that $\rho(M_Y) > 1$. Now, let us consider only particles in Y and neglect all the others. We obtain a supercritical multi-type Galton-Watson process with first moments M_Y that survives with positive probability since $\rho(M_Y) > 1$, compare with [5]. The subset Y takes the position of *recurrent seeds* in [3]. In contrast to [3] we don't need to construct the seeds explicitly but use Equation (8) in order to make the criterion explicit for branching random walk in random environment.

2.1 The critical BMC is transient

We have already mentioned in the introduction that the model we study in this paper is more general than the model where branching and movement are independent. Let us consider a BMC (X,P,μ) with independent branching and movement. Here (X,P) is an irreducible and infinite Markov chain in discrete time and

$$\mu(x) = (\mu_k(x))_{k\geq 1}$$

is a sequence of non-negative numbers satisfying

$$\sum_{k=1}^{\infty} \mu_k(x) = 1 \text{ and } m(x) := \sum_{k=1}^{\infty} k\mu_k(x) < \infty.$$

We define the BMC (X,P,μ) with underlying Markov chain (X,P) and branching distribution $\mu = (\mu(x))_{x\in X}$ following [8]. At time 0 we start with one particle in an arbitrary starting position $x \in X$. At each time each particle in position x splits up according to $\mu(x)$ and the offspring particles move according to (X,P). At any time, all particles move and branch independently of the other particles and the previous history of the process. Now, let $\boldsymbol{\omega}$ be a combination of multi-nomial distributions,

$$\omega_x(v) := \sum_k \mu_k(x) Mult\left(k; p(x,y), y \in \{y : p(x,y) > 0\}\right),$$

and hence $(X,\boldsymbol{\omega})$ has the same distribution as (X,P,μ). We immediately obtain the result of Theorem 3.2 in [4].

Theorem 2.5. *A BMC (X, P, μ) with constant mean offspring, i.e. $m(x) = m\ \forall x$, is transient if and only if $m \leq 1/\rho(P)$.*

The general (discrete) BMC can be used to study certain continuous-time branching random walks. Let us consider the branching random walk studied for example in [12], [10] and [14]. Let $G = (V, E)$ be a locally bounded and connected graph with vertex set V and edges E. The branching random walk (G, λ) on the graph G is a continuous-time Markov process whose state space is a suitable subset of $\mathbb{N}^V$. The process is described through the number $\eta(t, v)$ of particles at vertex v at time t and evolves according the following rules: for each $v \in V$

$$\begin{aligned} \eta(t,v) &\rightarrow \eta(t,v) - 1 \text{ at rate } \eta(t,v) \\ \eta(t,v) &\rightarrow \eta(t,v) + 1 \text{ at rate } \lambda \sum_{u: u\sim v} \eta(t,u), \end{aligned}$$

where λ is a fixed parameter and $u \sim v$ denotes that u is a neighbor of v. In words, each particle dies at rate 1 and gives birth to new particles at each neighboring vertex at rate λ. Let $o \in V$ be some distinguished vertex of G and denote $\mathbb{P}_o$ the probability measure of the process started with one particle at o at time 0. One says the branching random walk (G, λ) survives locally if

$$\mathbb{P}_o\left(\eta(t,o) = 0 \text{ for sufficiently large } t\right) < 1$$

or equivalently

$$\mathbb{P}_o\left(\exists (t_n)_{n\in\mathbb{N}} :\ t_n \to \infty \text{ and } \eta(t_n, o) > 0\right) = 0.$$

Since this is equivalent to the fact that the probability that infinitely many particles jump to o is zero, the question of local survival can be answered with an appropriate BMC. To this end let $(V, \boldsymbol{\omega})$ be any BMC with mean substitution $m(u,v) := \lambda$ if $u \sim v$ and $m(u,v) := 0$ otherwise. Hence, $M = \lambda \cdot A$, where A is the adjacency matrix of the graph $G = (V, E)$. Theorem 2.4 gives that the BMC $(V, \boldsymbol{\omega})$ is transient if and only if $\lambda \leq 1/\rho(A)$.

It is now straightforward to obtain the following result that strengthen Proposition 2.5 and Lemma 3.1 in [10] where the behavior in the critical case, $\lambda = 1/\rho(A)$, was not treated.

Corollary 2.6. *The branching random walk (G, λ) survives locally if and only if $\lambda > 1/\rho(A)$.*

3 BRWRE on $\mathbb{Z}^d$

We turn now to branching random walk in random environment (BRWRE) on $\mathbb{Z}^d$ and see how the results of the preceding section apply to this model. First, we define the model.

3.1 The model

Let $\mathfrak{U} \subset \mathbb{Z}^d$ be a finite generator of the group $\mathbb{Z}^d$. Define

$$\mathcal{V} := \left\{ v = (v_y,\ y \in \mathfrak{U}) : v_y \in \mathbb{N},\ \sum_{y\in\mathfrak{U}} v_y \geq 1 \right\}.$$

Furthermore, let us define the probability space that describes the random environment. To this end, let

$$\mathcal{M} := \left\{ \omega = (\omega(v),\ v \in \mathcal{V}) :\ \omega(v) \geq 0 \text{ for all } v \in \mathcal{V},\ \sum_{v \in \mathcal{V}} \omega(v) = 1 \right\}.$$

be the set of all probability measures ω on $\mathcal{V}$ and let Q be a probability measure on $\mathcal{M}$. For each $x \in \mathbb{Z}^d$ a random element $\omega_x \in \mathcal{M}$ is chosen according to Q independently. Let Θ be the corresponding product measure with one-dimensional marginal Q and denote $\mathcal{K} := supp(Q)$ the support of the marginal. The collection $\boldsymbol{\omega} = (\omega_x,\ x \in \mathbb{Z}^d)$ is called a realization of the random environment Θ. Each realization $\boldsymbol{\omega}$ defines a BMC $(\mathbb{Z}^d, \boldsymbol{\omega})$ and we denote $\mathbb{P}_{\boldsymbol{\omega}}$ the corresponding probability measure. Throughout the paper we will assume the following condition on Q that ensures the irreducibility of our process:

$$Q\left\{ \sum_y \omega_0(y) > \varepsilon \quad \forall y \in \mathfrak{G} \right\} = 1 \qquad \text{for some } \varepsilon > 0, \tag{9}$$

where $\mathfrak{G} \subset \mathfrak{U}$ is some generating set of $\mathbb{Z}^d$. For example, we can choose $\mathfrak{G} := \{\pm e_i, 1 \leq i \leq d\}$, where e_i is the ith coordinate vector of $\mathbb{Z}^d$. The uniform condition in (9) is used for Lemma 3.4 and Lemma 3.5 where we need that the BMC $(\mathbb{Z}^d, \boldsymbol{\sigma})$ is irreducible for all realization $\boldsymbol{\sigma} = (\sigma_x)_{x \in \mathbb{Z}^d}$ with $\sigma_x \in \mathcal{K}$.

3.2 Transience or strong recurrence

Due to condition (9) Θ-almost every $\boldsymbol{\omega}$ defines an irreducible matrix $M_{\boldsymbol{\omega}}$. Hence, with Lemma 2.2

$$\rho(M_{\boldsymbol{\omega}}) = \limsup_{n \to \infty} \left(m_{\boldsymbol{\omega}}{}^{(n)}(x,x) \right)^{1/n}, \quad \forall x \in \mathbb{Z}^d.$$

Furthermore, we have that the translation $\{T_z :\ z \in \mathbb{Z}^d\}$ acts ergodically as a measure preserving transformation on our environment. Together with the fact that $\limsup_{n \to \infty} \left(m_{\boldsymbol{\omega}}{}^{(n)}(x,x) \right)^{1/n}$ does not depend on x this implies that $\log \rho(M_{\boldsymbol{\omega}})$ is equal to a constant for Θ-a.a. realizations $\boldsymbol{\omega}$. Eventually, $\rho = \rho(M_{\boldsymbol{\omega}})$ for Θ-a.a. realizations $\boldsymbol{\omega}$ and some ρ, that we call the spectral radius of the BRWRE. Together with Theorem 2.4 this immediately implies that the BRWRE is either transient for Θ-a.a. realizations or recurrent for Θ-a.a. realizations. We have even the stronger result, compare with [3]:

Theorem 3.1. *We have either*

- *for Θ-a.a. realizations $\boldsymbol{\omega}$ the BRWRE is strongly recurrent:*

$$\alpha(\boldsymbol{\omega}, x) := \mathbb{P}_{\boldsymbol{\omega},x} \left(\sum_{n=1}^{\infty} \sum_{i=1}^{\eta(n)} \mathbf{1}\{x_i(n) = x\} = \infty \right) = 1 \quad \forall x \in X, \ \textit{or}$$

- *for Θ-a.a. realizations $\boldsymbol{\omega}$ the BRWRE is transient:*

$$\mathbb{P}_{\boldsymbol{\omega},x}\left(\sum_{n=1}^{\infty}\sum_{i=1}^{\eta(n)}\mathbf{1}\{x_i(n)=x\}=\infty\right)=0 \quad \forall x\in X.$$

Proof. It remains to show that $\rho>1$ implies $\alpha(\boldsymbol{\omega},x)=1$ for all $x\in X$. For Θ-a.a. $\boldsymbol{\omega}$ there exists some $Y\subset\mathbb{Z}^d$ such that $\rho(M_{\boldsymbol{\omega}_Y})>1$ and the corresponding multi-type Galton-Watson process is supercritical and survives with positive probability. We start the process in $x\in X$. Since the random environment is iid, it is easy to construct a sequence of independent multi-type Galton-Watson processes whose extinction probability are bounded away from 1, we refer to [3] and [9] for more details. At least one of these processes survives and infinitely many particles visit the starting position x, i.e. $\alpha(\boldsymbol{\omega},x)=1$. □

3.3 The spectral radius of BRWRE and the transience criterion

We first give the transience criterion.

Theorem 3.2. *The BRWRE is transient for Θ-a.a. realizations if*

$$\sup_{m\in\hat{\mathcal{K}}}\inf_{\theta\in\mathbb{R}^d}\left(\sum_s e^{\langle\theta,s\rangle}m(s)\right)\leq 1. \tag{10}$$

Otherwise it is strongly recurrent for Θ-a.a. realizations.

Remark 3.1. The fact that condition (10) is equivalent to *Condition L* of [3] follows through straightforward calculation and the fact that the sup and inf in (10) are attained.

The remaining part of the paper is devoted to the identification of ρ in order to show the explicit criterion for transience, Theorem 3.2. The first observation is that the spectral radius is *as large as possible*, compare with Lemma 3.3, then that it only depend on the convex hull of the support $\mathcal{K}$, compare with Lemma 3.4, and the last one that it equals the spectral radius of an appropriate homogeneous BMC, compare with Lemma 3.5.

The first Lemma is a straightforward generalization of [9] or alternatively follows from the proof of Lemma 3.4.

Lemma 3.3. *We have*

$$\rho:=\rho(M_{\boldsymbol{\omega}})=\sup_{\boldsymbol{\sigma}}\rho(M_{\boldsymbol{\sigma}}) \text{ for } \Theta\text{-a.a. } \boldsymbol{\omega},$$

where the sup *is over all possible collections $\boldsymbol{\sigma}=(\sigma_x)_{x\in\mathbb{Z}^d}$ with $\sigma_x\in\mathcal{K}$.*

Furthermore, the spectral radius does only depend on the convex hull $\hat{\mathcal{K}}$ of the support $\mathcal{K}$ of Q, compare with [17] where this is done for random walk in random environment.

Lemma 3.4. *We have*

$$\rho:=\rho(M_{\boldsymbol{\omega}})=\sup_{\hat{\boldsymbol{\sigma}}}\rho(M_{\hat{\boldsymbol{\sigma}}}) \text{ for } \Theta\text{-a.a. } \boldsymbol{\omega},$$

where the sup *is over all possible collections $\hat{\boldsymbol{\sigma}}=(\hat{\sigma}_x)_{x\in\mathbb{Z}^d}$ with $\hat{\sigma}_x\in\hat{\mathcal{K}}$.*

Proof. In order to see this recall that for any irreducible kernel K we have, due to Lemma 2.3 and Equation (8),

$$\rho(K) = \min\{t > 0 : \exists f(\cdot) > 0 \text{ such that } Kf \leq tf\}$$

and

$$\rho(K) = \sup_{Y} \rho(K_Y), \tag{11}$$

where the supremum is over finite and irreducible subsets $Y \subset \mathbb{Z}^d$. Let $F \subset \mathbb{Z}^d$ be an irreducible subset with respect to M and define as usual the complement $F^c := \mathbb{Z}^d \setminus F$, the boundary $\partial F := \{x \in F : \ m(x,y) > 0 \text{ for some } y \in F^c\}$ and the inner points $F^\circ = F \setminus \partial F$ of the set F. The key point is now to consider the equation

$$\sum_{y} m(x,y) f(y) \quad = \quad t \cdot f(x) \quad \forall x \in F^\circ \tag{12}$$

with boundary condition $f(y) = 1 \quad \forall y \in \partial F$. In order to give a solution of equation (12), we consider a modified process where we identify the border of F with a single point, say $\triangle$. Let $\widetilde{M}$ be the finite kernel over $\widetilde{F} := F^\circ \cup \{\triangle\}$ with $\widetilde{m}(x,y) := m(x,y)$ for all $x, y \in F^\circ$, $\widetilde{m}(x,\triangle) := \sum_{y \in \partial F} m(x,y)$ for all $x \in F^\circ$ and $\widetilde{m}(\triangle, x) = 0$ for all $x \in \widetilde{F}$. Observe that $\widetilde{M}$ is finite with absorbing state $\triangle$. Then the function

$$\widetilde{f}_F(t,x) := \sum_{k=0}^{\infty} \widetilde{m}^{(k)}(x,\triangle)(1/t)^k$$

is the unique solution of

$$\sum_{y} \widetilde{m}(x,y) f(y) \quad = \quad t \cdot f(x) \quad \forall x \in F^\circ \tag{13}$$

with boundary condition $f(\triangle) = 1$.

One can think of $\widetilde{f}_F(t,x)$ as the expected number of particles visiting ∂F for the first time in their ancestry line in the multi-type Galton-Watson process with first mean $1/t \cdot \widetilde{M}$ and one original particle at $x \in F$. Furthermore, we have that

$$\rho(\widetilde{M}) = \inf\{t > 0 : \ \widetilde{f}_F(t,x) < \infty \quad \forall x \in F\},$$

since $R(\widetilde{M}) = 1/\rho(\widetilde{M})$ is the convergence radius of $\widetilde{G}(x,x|z) := \sum_{k=0}^{\infty} \widetilde{m}^{(k)}(x,x) z^k$ for all $x \in F^\circ$. Since $m_F(x,y) = m(x,y)$ for all $x, y \in F$ we have that the convergence radius of $\widetilde{G}(x,x|z)$ equals the one of $G_F(x,x|z) := \sum_{k=0}^{\infty} m_F^{(k)}(x,x) z^k$ for all $x \in F^\circ$. Eventually,

$$\rho(M_F) = \inf\{t > 0 : \ f_F(t,x) < \infty \quad \forall x \in F\},$$

where $f_F(t,x) := \tilde{f}_F(t,x)$ for all $x \in F^\circ$ and $f_F(t,y) := \tilde{f}_F(t,\triangle)$ for all $y \in \partial F$.

The last step is now to determine for every finite set F the largest spectral radius over all possible choices of M_F with $m(x,\cdot) \in \mathcal{K}$ and show that this value does not change if we maximize

over all possible choices M_F with $m(x,\cdot) \in \hat{\mathcal{K}}$. We consider the following dynamical programming problem:

$$\sup_{m(x,\cdot)\in\mathcal{K}} \sum_y m(x,y) f(y) \quad = \quad t \cdot f(x) \quad \forall x \in F^\circ \tag{14}$$

with boundary condition $f(y) = 1 \quad \forall y \in \partial F$. The goal of the optimization problem is to maximize $f_F(t,x)$ over the possible choices of M_F. Observe that there will be a maximal value of t^* such that the solution f^* of the optimization problem is finite for all $t > t^*$. The value t^* is now equal to the largest spectral radius $\rho(M_F)$ we can achieve. We conclude with the observation, that t^* does not change if we replace $\mathcal{K}$ by its convex hull $\hat{\mathcal{K}}$ and the fact that $\rho(M) = \sup_F \rho(M_F)$. □

The next step is to show that the spectral radius ρ of the BRWRE equals the spectral radius of some homogeneous BMC and can therefore be calculated explicitly. We generalize the argumentation of [9] and [17].

Lemma 3.5. *For a RWRE on $\mathbb{Z}^d$ we have for Θ-a.a. realizations $\boldsymbol{\omega}$*

$$\begin{aligned} \rho(M_{\boldsymbol{\omega}}) &= \sup_{m\in\hat{\mathcal{K}}} \rho(M_m^h) && (15)\\ &= \sup_{m\in\hat{\mathcal{K}}} \inf_{\theta\in\mathbb{R}^d} \left(\sum_{s\in\mathfrak{U}} e^{\langle\theta,s\rangle} m(s) \right), && (16) \end{aligned}$$

where M_m^h is the transition matrix of the BMC with $m(x,x+s) = m(0,s) =: m(s)$ for all $x \in \mathbb{Z}^d,\ s \in \mathfrak{U}$.

Proof. The second equality is more or less standard. It follows for example from the fact that $\rho(P) = \exp(-I(0))$, where $I(\cdot)$ is the rate function of the large deviations of the random walk on $\mathbb{Z}^d$ with transition probabilities $P := M/\sum_s m(s)$. Since, due to Lemma 3.4, we have $\rho(M_{\boldsymbol{\omega}}) \geq \sup_{m\in\hat{\mathcal{K}}} \rho(M_m^h)$, it remains to show

$$\rho(M_{\boldsymbol{\omega}}) \leq \sup_{m\in\hat{\mathcal{K}}} \inf_{\theta\in\mathbb{R}^d} \left(\sum_s e^{\langle\theta,s\rangle} m(s) \right)$$

for Θ-a.a. realizations $\boldsymbol{\omega}$. Observing that the function $\phi(m(\cdot),\theta) := \left(\sum_s e^{\langle\theta,s\rangle} m(s)\right)$ is convex in θ and linear in $m(\cdot)$, we get by a standard minimax argument, compare with [13], that

$$\sup_{m\in\hat{\mathcal{K}}} \inf_{\theta\in\mathbb{R}^d} \sum_s e^{\langle\theta,s\rangle} m(s) = \inf_{\theta\in\mathbb{R}^d} \sup_{m\in\hat{\mathcal{K}}} \sum_s e^{\langle\theta,s\rangle} m(s) =: c.$$

Let $\varepsilon > 0$ and $\theta \in \mathbb{R}^d$ such that

$$\sup_{m\in\hat{\mathcal{K}}} \sum_s e^{\langle\theta,s\rangle} m(s) \leq c(1+\varepsilon).$$

By induction we have for any realization $\boldsymbol{\omega}$:

$$\sum_{x_n \in \mathbb{Z}^d} e^{\langle \theta, x_n \rangle} m_{\boldsymbol{\omega}}(0, x_n) \leq (c(1+\varepsilon))^n.$$

Therefore by observing only $x_n = 0$:

$$m_{\boldsymbol{\omega}}^{(n)}(0,0) \leq (c(1+\varepsilon))^n,$$

and hence $\rho(M_{\boldsymbol{\omega}}) \leq c(1+\varepsilon)$ for all $\varepsilon > 0$. $\square$

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